UNIVERSITY OF
LINCOLN

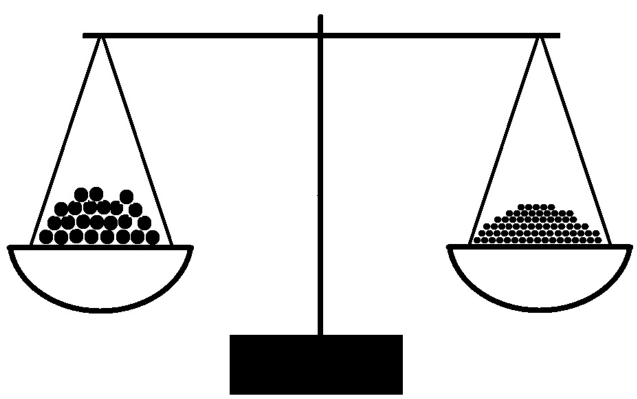
SCHOOL OF MATHEMATICS AND PHYSICS

Mathematics Challenge–2023–24

Brief solutions

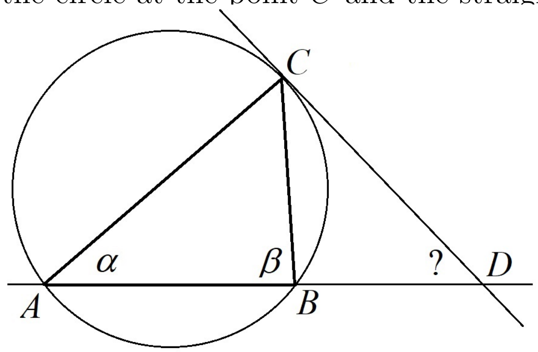
Note that each problem may have several different solutions by various methods.

Problem 1. The two bowls of a balance scale contain 2024 iron balls of two sizes, but each bowl contains only balls of the same size. The scales are in equilibrium. If 53 balls were to be taken off the left bowl, then transferring 100 balls from the right bowl into the left bowl would restore the equilibrium. How many balls are in each bowl?



Solution of Problem 1. If l is the weight of a 'left' ball, and r of a 'right' ball, then $53l = 200r$, whence $l = 200r/53$. Let x be the number of 'left' balls. Then $xl = (2024 - x)r$, whence by substituting we get $x \cdot 200r/53 = (2024 - x)r$; $200x = 53 \cdot (2024 - x)$; $x = 424$ balls (on the left bowl), and $2024 - 424 = 1600$ balls on the right bowl.

Problem 2. A triangle ABC is inscribed in a circle. Given $\angle CAB = \alpha$ and $\angle CBA = \beta$, find the angle between the tangent to the circle at the point C and the straight line through A and B.



Solution of Problem 2. By the alternate segment theorem, $\angle BCD = \alpha$. Hence $\angle BDC = 180° - \alpha - (180° - \beta) = \beta - \alpha$.

Comments on submissions and solutions of Problem 2. Some contestants also included a proof of the alternate segment theorem, although this was not required and proofs simply referring to this theorem were given full marks. One can also correctly note that the answer is $\alpha - \beta$ if $\beta < \alpha$, contrary to the picture in the question, and that there is no intersection point D if $\alpha = \beta$, when the tangent is parallel to AB.

Problem 3. Find all positive integers x, y, z satisfying the equation $2^x + 3^y = z^2$.

Solution of Problem 3. We use the following notation: $\mathbb{N}$ is the set of positive integers; $a \mid b$ means that an integer b is divisible by an integer a, and $a \nmid b$ if not; we also write $a \equiv b \pmod{c}$ if a and b have the same remainder after division by c, read as "a is congruent to b modulo c".

First note that $3 \nmid z$, since $3 \nmid 2^x$ as x is a positive integer. Then $z^2 \equiv 1 \pmod 3$, since $(3k+1)^2 \equiv 1 \pmod 3$ and $(3k+2)^2 \equiv 1 \pmod 3$ for $k \in \mathbb{N}$. Hence $2^x \equiv 1 \pmod 3$, which implies that x is even: we have $2^{2k} \equiv 4^k \equiv 1^k \equiv 1 \pmod 3$ and $2^{2k+1} \equiv 2 \cdot 2^{2k} \equiv 2 \cdot 1 \equiv 2 \pmod 3$.

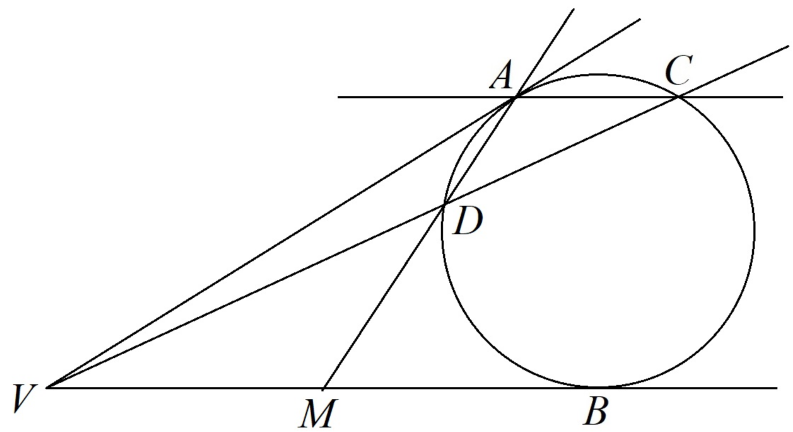
We write $x = 2k$ for $k \in \mathbb{N}$ and then factorize $3^y = z^2 - 2^{2k} = (z - 2^k)(z + 2^k)$. Each factor on the right must be a power of 3; write $z - 2^k = 3^l$ and $z + 2^k = 3^{y-l}$. Then the difference $(z + 2^k) - (z - 2^k) = 2^k + 2^k = 2^{k+1}$ is equal to $3^{y-l} - 3^l$, which is divisible by 3 unless $l = 0$. Since $3 \nmid 2^{k+1}$, we have $l = 0$.

Thus, $z - 2^k = 1$ and $z + 2^k = 3^y$, whence $2^k + 1 + 2^k = 3^y$, that is, $2^{k+1} + 1 = 3^y$. One can easily show that odd powers of 3 have remainder 3 modulo 4, and since $2^{k+1} + 1$ has remainder 1 modulo 4 (as $k \in \mathbb{N}$), we must have $y = 2n$ even.

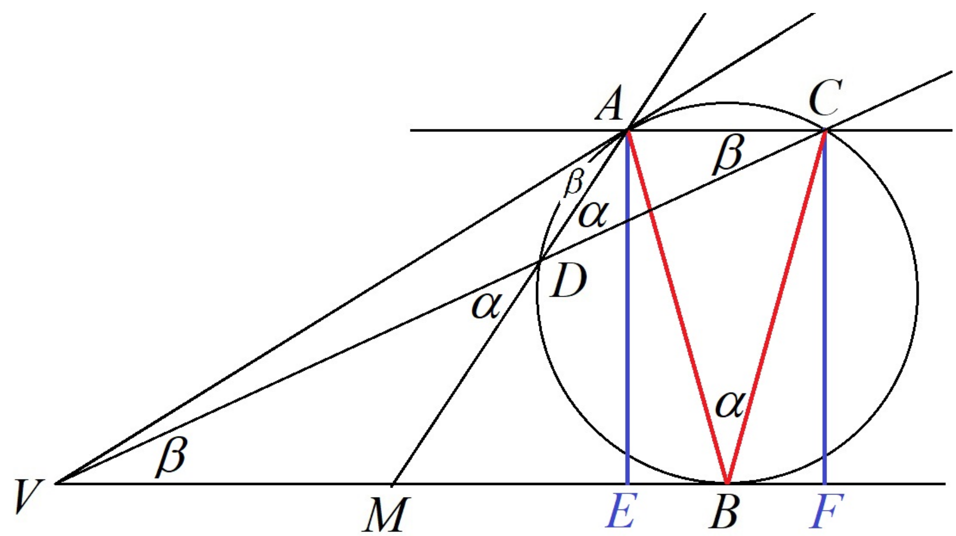
Then $2^{k+1} = 3^{2n} - 1 = (3^n - 1)(3^n + 1)$. Each factor on the right must be a power of 2, while the difference between them is 2. This clearly only happens for 2 and 4, so that $3^n - 1 = 2$ and $3^n + 1 = 4$. As a result, $n = 1$, $y = 2$, $2^{k+1} = 8$, $k = 2$, $x = 4$, and then from $2^4 + 3^2 = z^2$ we get $z = 5$.

Comments on submissions and solutions of Problem 3. One can arrive at the same conclusion using other divisibility considerations, as was indeed achieved in some submissions. However, simply guessing the correct answer 4, 2, 5 is not nearly enough without proof that there are no other solutions. The fact that 2 and 4 are the only powers of 2 (with positive integer exponents) that have difference 2 was also proved in some submissions, but assuming this fact as obvious was allowed for getting full marks if other arguments were correct.

Problem 4. A circle is inscribed in an angle with vertex V with tangency points A, B. The straight line parallel to VB passing through A intersects the circle at another point C. The straight line through V and C intersects the circle at another point D. The straight line through A and D intersects the line VB at a point M. Prove that M is the middle point of the segment VB.



Solution of Problem 4. Since $VA = VB$, we need to show that $VM : VA = 1 : 2$. By the sine theorem ('sine rule'), we have $VM : \sin\angle VAM = VA : \sin\angle VMA$; therefore all we need to show is that $\sin\angle VAM : \sin\angle VMA = 1 : 2$. We know that $\sin\angle VMA = \sin\angle BMA$, since $\angle VMA = 180° - \angle BMA$, and that $\angle BMA = \angle AVM + \angle VAM$. We also know that $\angle VAM = \angle ACV$ by the alternate segment theorem, and $\angle ACV = \angle CVM$. To lighten the notation, let $\alpha = \angle AVM$ and $\beta = \angle CVM$. Then our task is to show that $\sin(\alpha + \beta) = 2\sin\beta$. Using the sine of the sum formula, we transform this to $\sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta = 2\sin\beta$, and then divide by $\cos\beta$ to obtain $\sin\alpha + \cos\alpha \cdot \tan\beta = 2\tan\beta$, or equivalently, $\sin\alpha = \tan\beta \cdot (2 - \cos\alpha)$.



We now express $\sin\alpha$, $\cos\alpha$, $\tan\beta$ in terms of $b = AB = CB$ and $l = VA = VB$ using the right triangles AVE and CVF, where E and F are the bases of perpendiculars dropped from A and C onto the line VB.

But first we note that $\angle ADC = \angle VDM = \angle AMB - \beta = \alpha$ and therefore also $\angle ABC = \angle ADC = \alpha$. Hence the equilateral triangles AVB and ABC are similar, and we can find AC from the proportion $AC : b = b : l$, that is, $AC = b^2/l$. Clearly, $EB = BF = AC/2 = b^2/2l$, and then $VE = l - b^2/2l$ and $VF = l + b^2/2l$.

We now can find $AE = CF = \sqrt{b^2 - (b^2/2l)^2}$ by Pythagoras, but in fact we do not need this expression, just denote $h = AE = CF$. Then $\tan\beta = \dfrac{h}{l + b^2/2l}$ from $\triangle CVF$. We also have $\cos\alpha = \dfrac{l - b^2/2l}{l}$ and $\sin\alpha = \dfrac{h}{l}$ from $\triangle AVE$. Substituting into the required equation $\sin\alpha = \tan\beta \cdot (2 - \cos\alpha)$ we see that it holds true:

$$\frac{h}{l} = \frac{h}{l + b^2/2l} \cdot \left(2 - \frac{l - b^2/2l}{l}\right);$$
$$\frac{h}{l} = \frac{h}{l + b^2/2l} \cdot \left(\frac{2l - l + b^2/2l}{l}\right);$$
$$\frac{h}{l} = \frac{h}{l + b^2/2l} \cdot \left(\frac{l + b^2/2l}{l}\right).$$

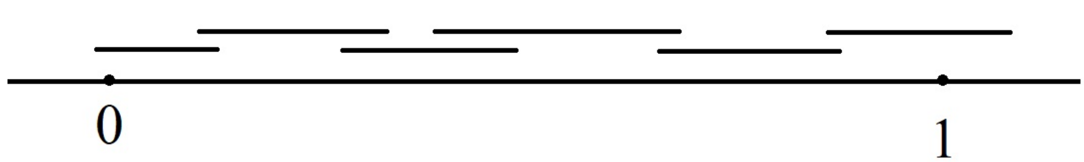
Comments on submissions and solutions of Problem 4. One can also prove the required result using the 'method of coordinates', that is, writing the equations of the circle, its tangents, finding the coordinates of intersection points, etc.; such solutions were also given full marks. One does wonder if there is also some 'more geometrical' solution, without much of the calculations.

Problem 5. The real line segment $0 \leqslant x \leqslant 1$ is completely covered with 2024 intervals of various lengths, possibly with overlappings. Prove that it is always possible to choose some of these intervals in such a way that the sum of the lengths of the chosen intervals is at least 0.5 and any two of the chosen intervals are disjoint.

Solution of Problem 5. It is more natural to prove the same result with an arbitrary number n of intervals covering the segment $[0, 1]$. If there is an interval that is also completely covered by the other intervals, then this interval can be discarded, as the others also completely cover $[0, 1]$ and we could choose the required pairwise disjoint ones with total length $\geqslant 0.5$ among these others. (The same argument can be formalized as indiction on n: when $n = 1$ the result is clear; suppose that the result is already proved for $n = k$, then for $k + 1$ intervals, if one of them is completely covered by the other intervals, then we apply the induction hypothesis to these other k intervals and obtain the required pairwise disjoint family with total length $\geqslant 0.5$.)

We can also assume that all the intervals are closed, that is, include their endpoints. Indeed, if we add all endpoint to some possibly open or semi-closed intervals, then the resulting intervals will of course also completely cover $[0, 1]$, and if we succeed in choosing pairwise disjoint closed intervals with total length $\geqslant 0.5$, then the intervals from which these were possibly obtained by adding endpoints would also be disjoint with the same total length.

Therefore from now on we assume that neither of the intervals is completely covered by the other intervals, and that all intervals contain their endpoints. We claim that then the intervals form a chain like on this picture, where intervals have common parts with one another only in consecutive pairs, without triple common parts, so that alternate intervals are disjoint:

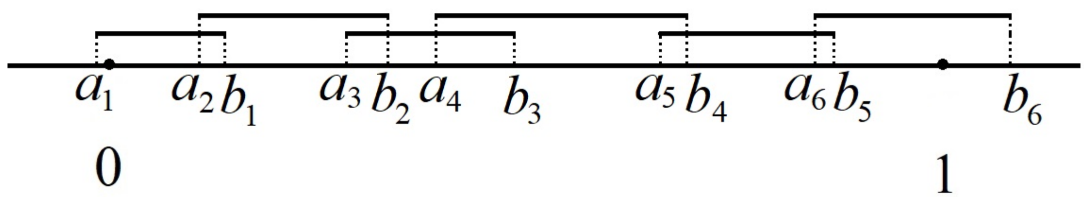


Here the intervals are depicted sightly raised above the real line for better visibility, and alternate intervals are at the same level.

To better formulate and rigorously prove this claim, let us numerate the intervals in the order in which their left endpoints appear on the real line from left to right, and let a_k and b_k be the left and right endpoints of the k-th interval. Note that our intervals must have different left endpoints, since otherwise one would cover the other, contrary to our assumption. Thus, we have strict inequalities $a_1 < a_2 < a_3 < \cdots$. We claim that, moreover, we have the inequalities

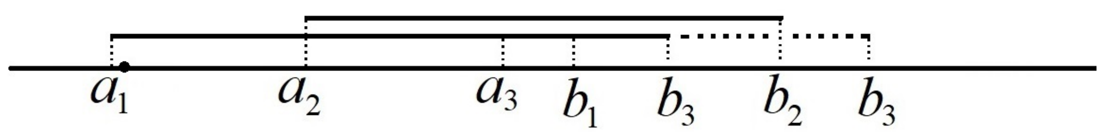
$$a_1 < a_2 \leqslant b_1 < a_3 \leqslant b_2 < a_4 \leqslant b_3 < a_5 \leqslant \cdots ,$$

as in this example with $n = 6$:



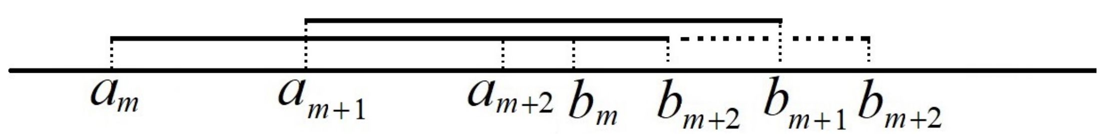
In other words, for our n intervals we prove by induction that $a_{k+1} \leqslant b_k < a_{k+2}$ for every $k = 1, 2, \ldots, n-2$. In particular, the strict inequalities $b_k < a_{k+2}$ will mean that the intervals with odd indices are pairwise disjoint, as well as that the intervals with even indices are pairwise disjoint. We also need the inequalities $a_{k+1} \leqslant b_k$ for the argument to work by induction.

At the base of induction $k = 1$, we need to show that $a_2 \leqslant b_1 < a_3$. If we had $a_2 > b_1$, then the points of $[0, 1]$ between b_1 and a_2 would not be covered at all; hence $a_2 \leqslant b_1$. If we had $b_1 \geqslant a_3$, then one of the intervals $[a_2, b_2]$ or $[a_3, b_3]$ would be covered by other intervals, depending on whether $b_3 \leqslant b_2$ or $b_3 \geqslant b_2$ (we already know that $a_2 \leqslant b_1$):



Here two possibilities for b_3 are shown on the same picture. This contradicts our assumption that no interval is covered by the others. Hence $b_1 < a_3$.

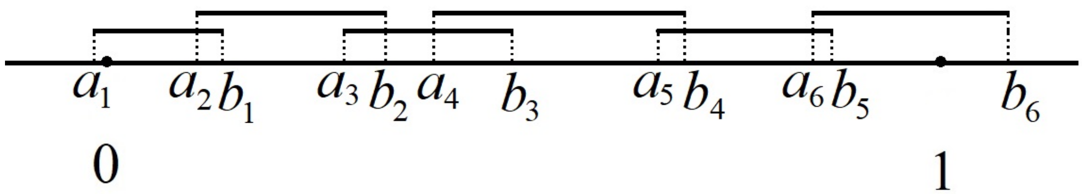
Now suppose that the inequalities $a_{k+1} \leqslant b_k < a_{k+2}$ hold for all $k < m < n-2$; we now prove that $a_{m+1} \leqslant b_m < a_{m+2}$. If we had $a_{m+1} > b_m$, then, in view of the previous inequalities $a_1 < a_2 \leqslant b_1 < a_3 \leqslant \cdots b_2 < a_4 \leqslant b_3 < a_5 \leqslant \cdots a_m \leqslant b_{m-1} < a_{m+1}$ assumed by induction, the points of $[0, 1]$ between b_m and a_{m+1} would not be covered at all; hence $a_{m+1} \leqslant b_m$. If we had $b_m \geqslant a_{m+2}$, then one of the intervals $[a_{m+1}, b_{m+1}]$ or $[a_{m+2}, b_{m+2}]$ would be covered by other intervals (depending on whether $b_{m+2} \leqslant b_{m+1}$ or $b_{m+2} \geqslant b_{m+1}$ (we already know that $a_{m+1} \leqslant b_m$):



Here two possibilities for b_{m+2} are shown on the same picture. This contradicts our assumption that no interval is covered by the others. Hence $b_m < a_{m+2}$.

Thus, we have established the inequalities

$$a_1 < a_2 \leqslant b_1 < a_3 \leqslant b_2 < a_4 \leqslant b_3 < a_5 \leqslant \cdots$$



The strict inequalities $b_k < a_{k+2}$ mean that the intervals with odd indices are pairwise disjoint, as well as that the intervals with even indices are pairwise disjoint. Their total length is at least 1, so one of these sets of intervals (with even indices, or with odd indices) must have total length at least 0.5.

Comments on submissions and solutions of Problem 5. Clearly, after removing (say, one by one) the intervals that are completely covered by the others, the remaining intervals do completely cover $[0, 1]$. It is tempting to immediately claim that then 'obviously' the intervals with odd indices are pairwise disjoint, as well as that the intervals with even indices are pairwise disjoint, and then the result follows as at the end of the solution above. But this is not that obvious, and a rigorous proof was required. Without such a proof, only partial marks were given to such solutions. The finer point about endpoint of the intervals was not taken into account, so if a submitted solution assumed that all intervals were closed (or all open), no marks were deducted.

Problem 6. Given a table 7×7, in how many ways can one fill it with numbers 0 and 1 so that the following two conditions simultaneously hold:

(a) the sums over the 7 rows are all different, and
(b) the sums over the 7 columns are all different?

Solution of Problem 6. There are 8 possible sums for each row or column: $0, 1, \ldots, 7$. Therefore exactly one of these numbers must be missing among the sums over rows and among the sums over columns. Since the sum of the sums over rows is equal to the sum of the sums over columns (as this is the sum of all entries), the missing number is the same for rows and for columns. This missing number can only be 0 or 7. Indeed, otherwise there is, say, a row with all 1s, and a column with all 0s, which is impossible in view of their intersection. For every arrangement of missing 0 type there is a complementary arrangement of missing 7 type, when all 0s are replaced by 1s and all 1s by 0s. Every arrangement is obviously of only one type. Therefore the total number of arrangements is double the number of arrangements of a single type.

Consider for definiteness arrangements of missing 0 type. Note that such do exist: for example put 1s in the diagonal cells and in all cells above the diagonal. Let us call this the initial arrangement.

1	1	1	1	1	1	1
0	1	1	1	1	1	1
0	0	1	1	1	1	1
0	0	0	1	1	1	1
0	0	0	0	1	1	1
0	0	0	0	0	1	1
0	0	0	0	0	0	1

Every permutation of rows produces another 'good' arrangement. Different permutations produce different arrangements, since the order of row sums changes. The same applies to permutations of columns. Thus we obtain $7! \cdot 7!$ different arrangements for the given missing 0 type. Together with the same number of missing 7 type, this amounts to $2 \cdot 7! \cdot 7!$ arrangements with required properties.

It remains to prove that there are no other arrangements (say, of missing 0 type). For that it is sufficient to show that any good arrangement can be transformed by a permutation of columns and a permutation of rows to the initial arrangement; then the reverse permutations produce this good one from the initial one as described in the preceding paragraph. Just rearrange columns in such a way that the sums over the columns are $1, 2, 3, \ldots, 6, 7$ when read from left to right, and then rearrange rows in such a way that the sums over the rows are $7, 6, 5, \ldots, 2, 1$ when read from top to bottom. Then the 7th column is filled with 1s, the 6th column must have 0 at the bottom (and only 1s above it), since the only 1 in the bottom 7th row is in the 7th position, the 5th column must have two 0s at the bottom (and only 1s above), since the only 1 in the bottom 7th row is in the 7th position and the only two 1s in the 6th row are in 6th and 7th positions, and so on; this is the initial arrangement as required.

Comments on submissions and solutions of Problem 6. The explanations why there are no other arrangements were given to varying degree of rigour. But the problem was regarded as nearly completely solved if the existence of $2 \cdot 7! \cdot 7!$ arrangements was shown.